

TIME-CHANGES OF HOROCYCLE FLOWS

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ABSTRACT. We consider smooth time-changes of the classical horocycle flows on the unit tangent bundle of a compact hyperbolic surface and prove sharp bounds on the rate of equidistribution and the rate of mixing. We then derive results on the spectrum of smooth time-changes and show that the spectrum is absolutely continuous with respect to the Lebesgue measure on the real line and that the maximal spectral type is equivalent to Lebesgue.

1. INTRODUCTION

The classical horocycle flow is a fundamental example of a unipotent, parabolic (non-hyperbolic) flow. Its dynamical properties have been studied in great detail. It is known that the flow is minimal [10], uniquely ergodic [6], has Lebesgue spectrum and is therefore strongly mixing [16], in fact mixing of all orders [14], and has zero entropy [9]. Its finer ergodic and rigidity properties, as well as the rate of mixing, were investigated by M. Ratner in a series of papers [17], [18], [19], [20] (for results on the rate of mixing of the geodesic as well as horocycle flows see also the paper by C. Moore [15]). In joint work with L. Flaminio [5], the first author has proved precise bounds on ergodic integrals of smooth functions. In the case of finite-volume, non-compact surfaces, the horocycle flow is not uniquely ergodic and the classification of invariant measures is due to Dani [4]. The asymptotic behaviour of averages along closed horocycles has been studied by D. Zagier [24], P. Sarnak [21], D. Hejhal [11] and more recently in [5] and by A. Strömbergsson [22]. Horocycle flows on general geometrically finite surfaces have been studied by M. Burger [3].

Not much is known for general smooth parabolic flows, not even for smooth perturbations of classical horocycle flows in the compact case. In fact, even the dynamics of non-trivial smooth time-changes is poorly understood. Our paper addresses the latter question. By the classification of horocycle invariant distributions [5] and by the related results on asymptotic of ergodic averages for classical horocycle flows (see [5], [2]), it is known that smooth time-changes which are measurably trivial form a subspace of countable codimension, so that the generic smooth time-change is not even measurably conjugate to the horocycle flow. It is therefore interesting to know, perhaps as a step towards a better understanding of parabolic dynamical systems, to what extent the dynamical properties of the horocycle flow persist after a smooth time-change. The most important result to date is the proof by B. Marcus more than thirty years ago that all time-changes satisfying a mild differentiability conditions are mixing [14]. Marcus results generalized earlier work by Kushnirenko who proved mixing for all time-changes with sufficiently small derivative in the geodesic direction [13].

A. Katok and J.-P. Thouvenot have conjectured that “Any flow obtained by a sufficiently smooth time change from a horocycle flow has countable Lebesgue

spectrum" (see [12], Conjecture 6.8). In fact, the question on the spectral type of smooth time changes of horocycle flows was already asked in Kushnirenko's paper [13]. There the author is able to prove the relative absolute continuity of the spectrum of (restricted) smooth perturbations of skew-shifts, but cannot extend his results to time-changes of horocycle flows. In our paper we prove sharp bounds on the rate of equidistribution and mixing of smooth time-changes of the classical horocycle flow on the unit tangent bundle of a compact hyperbolic surface (see Theorem 2 and Theorem 3 in Sections 3 and 4 respectively). We then derive results on the spectrum of smooth time-changes (in Section 6), most notably we prove that the spectrum is absolutely continuous with respect to the Lebesgue measure (Theorem 6, in Section 6.2). We finally prove that the maximal spectral type is indeed equivalent to Lebesgue (Theorem 7, in Section 6.3).

The guiding idea of our work is that Marcus' mixing mechanism can be made quantitative by the more recent quantitative results on the rate of equidistribution for horocycle flows (see [3], [5], [2]). In fact, Marcus argument is based on the equidistribution of long *horocycle-like arcs*, that is, arcs which are long in the horocycle direction and bounded in the complementary directions. Sharp results on the rate of equidistributions of horocycle-like arcs were recently obtained in [2] as a refinement of earlier results for horocycle arcs [3], [5]. Finally, our estimates on the rate of mixing for time-changes would be far from optimal, and definitely too weak to derive any significant spectral result, without a key *bootstrap trick*. Thanks to this bootstrap trick we can prove that decay of correlations of the horocycle flows are indeed stable under any smooth time-change.

Spectral results are derived from square-mean bounds on twisted ergodic integrals of smooth functions which are equivalent to bounds on the Fourier transform of the spectral measures. A well-known difficulty in this approach is that the decay of correlations of a general smooth function under the horocycle flow is not square-integrable, so that it would seem hopeless to prove absolute continuity of the spectrum in this way. However, our results on decay of correlations of time-changes are precise enough, thanks to the bootstrap trick, to give optimal, and hence square-integrable, decay of correlations for smooth coboundaries. Once it is established that all smooth coboundaries have absolutely continuous spectral measures, it follows (for instance by a density argument) that the spectrum is purely absolutely continuous. Our estimates on decay of correlations of coboundaries are also crucial in the proof that the maximal spectral type is Lebesgue.

Let us remark that while in this paper we only deal with horocycle flows for compact hyperbolic surfaces, most of the methods and results can presumably be extended to the non-compact, finite volume case with appropriate modifications.

Structure of the paper. In Section 2 we introduce basic definitions, notation and properties of smooth time-changes of the classical horocycle flow. In Section 3 we recall the results on the invariant distributions for the classical horocycle flow from [5] and, from the results on the asymptotics of ergodic integrals in [5, 2], we derive analogous results for smooth-time changes (Theorem 2). These quantitative equidistribution results are used in Section 4 together with a key bootstrap trick to make quantitative Marcus' mixing argument for smooth-time changes and derive the quantitative mixing result in Theorem 3. In Section 5 we prove mean-square bounds on twisted ergodic integrals of smooth functions. Finally, in Section 6 we prove our main spectral results. We first prove a local estimate (Theorem 5 in

§ 6.1), then absolute continuity if the spectrum (Theorem 6 in § 6.2) and finally that the maximal spectral type is Lebesgue (Theorem 7 in § 6.3).

2. TIME-CHANGES OF HOROCYCLE FLOWS

Let $\{U, V, X\}$ the basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $PSL(2, \mathbb{R})$ given by the generators U and V of the stable and unstable horocycle flows and by the generator X of the geodesic flow, respectively. The following commutation relations hold:

$$(1) \quad [U, V] = 2X, \quad [X, U] = U, \quad [X, V] = -V.$$

Let $\{h_t^U\}$ and $\{h_t^V\}$ denote respectively the stable and unstable horocycle flows and let $\{\phi_t^X\}$ denote the geodesic flow on a compact homogeneous space $M := \Gamma \backslash PSL(2, \mathbb{R})$. They are defined respectively by the multiplicative action on the right of the 1-parameter subgroups of the group $PSL(2, \mathbb{R})$ listed below:

$$(2) \quad \{\exp(tU)\}_{t \in \mathbb{R}}, \quad \{\exp(tV)\}_{t \in \mathbb{R}}, \quad \{\exp(tX)\}_{t \in \mathbb{R}}.$$

A smooth time-change of the (stable) horocycle flow is a flow $\{h_t^\alpha\}$ on M defined as follows. Let $\tau : M \times \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cocycle over the flow $\{h_t^U\}$, that is, a function with the property that

$$\tau(x, t + t') = \tau(x, t) + \tau(h_t^U(x), t'), \quad \text{for all } (x, t, t') \in M \times \mathbb{R}^2.$$

We denote by $\alpha : M \rightarrow \mathbb{R}^+$ the infinitesimal generator of the cocycle $\tau : M \times \mathbb{R} \rightarrow \mathbb{R}$, that is, the function defined as follows:

$$\alpha(x) := \frac{\partial \tau}{\partial t}(x, t)|_{t=0}, \quad \text{for all } x \in M.$$

The time-change $\{h_t^\alpha\}$ is the flow on M generated by the smooth vector field

$$(3) \quad U_\alpha =: U/\alpha.$$

One can check that $\{h_t^\alpha\}$ is given by the formula

$$(4) \quad h_{\tau(x, t)}^\alpha(x) := h_t^U(x), \quad \text{for all } (x, t) \in M \times \mathbb{R}.$$

The flow $\{h_t^\alpha\}$ preserves the (smooth) volume form $\text{vol}_\alpha := \alpha \text{vol}$; in fact

$$\mathcal{L}_{U_\alpha} \text{vol}_\alpha = \iota_{U_\alpha} \text{vol}_\alpha = \iota_U \text{vol} = \mathcal{L}_U \text{vol} = 0.$$

We will assume below that the function $\alpha : M \rightarrow \mathbb{R}$ is everywhere strictly positive and is normalized so that

$$(5) \quad \int_M \text{vol}_\alpha = \int_M \alpha \text{vol} = 1.$$

The time-change $\{h_t^\alpha\}$ is parabolic, in fact the infinitesimal divergence of trajectories is at most quadratic with respect to time (as is the case for the standard horocycle flow). The tangent flow $\{Dh_t^\alpha\}$ on TM is described as follows.

Lemma 1. *The tangent flow $\{Dh_t^\alpha\}$ on TM is given by the following formulas:*

$$(6) \quad \begin{aligned} Dh_t^\alpha(V) &= \left[\int_0^t \left(\left(\int_0^\tau \frac{1}{\alpha} \circ h_u^\alpha du \right) \left(\frac{X\alpha}{\alpha} - 1 \right) \circ h_\tau^\alpha + \frac{V\alpha}{\alpha} \circ h_\tau^\alpha \right) d\tau \right] U_\alpha \circ h_t^\alpha \\ &\quad + V \circ h_t^\alpha + \left[\int_0^t \frac{1}{\alpha} \circ h_\tau^\alpha d\tau \right] X \circ h_t^\alpha; \\ Dh_t^\alpha(X) &= \left[\int_0^t \left(\frac{X\alpha}{\alpha} - 1 \right) \circ h_\tau^\alpha d\tau \right] U_\alpha \circ h_t^\alpha + X \circ h_t^\alpha; \end{aligned}$$

Proof. For any vector field W on M , let us write

$$(h_t^\alpha)_*(W) = a_t U_\alpha + b_t V + c_t X.$$

By the commutation relation

$$[U_\alpha, X] = \left(\frac{X\alpha}{\alpha} - 1\right)U_\alpha \quad \text{and} \quad [U_\alpha, V] = X/\alpha + \frac{V\alpha}{\alpha}U_\alpha,$$

hence

$$\frac{d}{dt}(h_t^\alpha)_*(W) = [c_t\left(\frac{X\alpha}{\alpha} - 1\right) \circ h_t^\alpha + b_t\frac{V\alpha}{\alpha} \circ h_t^\alpha]U_\alpha + \left(\frac{b_t}{\alpha} \circ h_t^\alpha\right)X.$$

It follows that the function (a_t, b_t, c_t) satisfies the following system of O.D.E.'s:

$$\begin{cases} \frac{da_t}{dt} &= c_t\left(\frac{X\alpha}{\alpha} - 1\right) \circ h_t^\alpha + b_t\frac{V\alpha}{\alpha} \circ h_t^\alpha; \\ \frac{db_t}{dt} &= 0; \\ \frac{dc_t}{dt} &= \frac{b_t}{\alpha} \circ h_t^\alpha. \end{cases}$$

If $W = V$, the initial condition is $(a_0, b_0, c_0) = (0, 1, 0)$, hence the unique solution of the Cauchy problem is given by the functions:

$$\begin{aligned} a_t &= \int_0^t \left(\int_0^\tau \frac{1}{\alpha} \circ h_u^\alpha du \right) \left(\frac{X\alpha}{\alpha} - 1 \right) \circ h_\tau^\alpha + \frac{V\alpha}{\alpha} \circ h_\tau^\alpha d\tau, \\ b_t &\equiv 1, \quad c_t = \int_0^t \frac{1}{\alpha} \circ h_\tau^\alpha d\tau. \end{aligned}$$

If $W = X$, the initial condition is $(a_0, b_0, c_0) = (0, 0, 1)$, hence the unique solution of the Cauchy problem is given by the functions:

$$a_t = \int_0^t \left(\frac{X\alpha}{\alpha} - 1 \right) \circ h_\tau^\alpha d\tau, \quad b_t \equiv 0, \quad c_t \equiv 1;$$

Formula (6) is therefore proved. \square

3. COHOMOLOGICAL EQUATION AND QUANTITATIVE EQUIDISTRIBUTION

It is a general fact that all properties of the cohomological equation and of the asymptotics of ergodic integrals for all smooth time-changes of any smooth flow can be read from the corresponding properties of the flow itself.

Let $L^2(M) := L^2(M, \text{vol})$ denote the Hilbert space of square-integrable function with respect to the standard volume form and, for any $r \geq 0$ let $W^r(M) \subset L^2(M)$ denote the standard Sobolev spaces on the compact manifold M and let $W^{-r}(M)$ denote the dual spaces.

Let $L^2(M, \text{vol}_\alpha)$ denote the Hilbert space of square-integrable function with respect to the $\{h_t^\alpha\}$ -invariant volume form and let

$$L_0^2(M, \text{vol}_\alpha) := \{f \in L^2(M) \mid \int_M f \text{vol}_\alpha = 0\}.$$

Let $\mathcal{D}'(M)$ be the space of distributions on M . For any distribution $D \in \mathcal{D}'(M)$, let D_α be the distribution defined as follows:

$$D_\alpha(f) := D(\alpha f), \quad \text{for all } f \in C^\infty(M).$$

The distribution D_α is well-defined and belongs to the dual Sobolev space $W^{-r}(M)$ whenever $\alpha \in W^r(M)$ and $D \in W^{-r}(M)$ for any $r > 3/2$. In fact, the Sobolev space $W^r(M)$, endowed with the standard structure of Hilbert space and with the

standard product of functions, is a Banach algebra for $r > \dim(M)/2$, which is here the case since M is a 3-dimensional manifold. The subspace $\mathcal{I}_\alpha^{-r}(M) \subset W^{-r}(M)$ of invariant distributions for the time-change $\{h_t^\alpha\}$ can be described in terms of the subspace $\mathcal{I}_U^{-r}(M) \subset W^{-r}(M)$ of invariant distributions for the horocycle flow (described in [5]).

Lemma 2. *Let $r > 3/2$ and let $\alpha \in W^r(M)$. The following holds:*

$$\mathcal{I}_\alpha^{-r}(M) := \{D_\alpha | D \in \mathcal{I}_U^{-r}(M)\}.$$

Proof. By the algebra property of the Sobolev space $W^r(M)$ for $r > 3/2$, for any non-vanishing function $\alpha \in W^r(M)$, the map $D \rightarrow D_\alpha$ is an automorphism of the Sobolev space $W^{-r}(M)$ (it is continuous, invertible with continuous inverse). By definition, a distribution $D_\alpha \in \mathcal{I}_\alpha^{-r}(M)$, that is, the distribution $D_\alpha \in W^{-r}(M)$ is invariant for the time-change $\{h_t^\alpha\}$ of generator $U_\alpha = U/\alpha$, if and only if

$$D_\alpha(U_\alpha f) = D_\alpha(U f / \alpha) = D(U f) = 0, \quad \text{for all } f \in W^{r+1}(M),$$

if and only if the distribution $D \in W^{-r}(M)$ is invariant under the horocycle flow $\{h_t^U\}$, that is, if and only if $D \in \mathcal{I}_U^{-r}(M)$, as stated. \square

Let us recall that we say that a function f on M is a coboundary for the flow $\{h_t^\alpha\}$ if there exists a function u on M , called the transfer function, such that $U_\alpha f = u$. The subspace of coboundaries for the time-changes of the horocycle flow is described by the following dictionary.

Lemma 3. *A function f on M is a coboundary for the time-change $\{h_t^\alpha\}$ with transfer function u on M if and only if the function αf is a coboundary for the flow $\{h_t^U\}$ with transfer function u on M .*

Proof. It follows immediately from the definition of coboundary recalling that by definition $U_\alpha = U/\alpha$. \square

By the above lemmas the theory of the cohomological equation for time-changes is reduced to that for the classical horocycle flow, developed in [5]. We state the main results below for the convenience of the reader.

By the theory of unitary representations for $SL(2, \mathbb{R})$ (see [1], [7], [8]), the Sobolev spaces $W^r(M)$ split as direct sums of irreducible sub-representations and each irreducible sub-representations characterized up to unitary equivalence by the spectral value $\mu \in \sigma(\square)$ of the restriction of the Casimir operator \square (a normalized generator of the center of the enveloping algebra), that is, for all $r \in \mathbb{R}$, the following splitting holds:

$$(7) \quad W^r(M) = \bigoplus_{\mu \in \sigma(\square)} W^r(H_\mu)$$

Non-trivial irreducible unitary representations belong to three different series: the *principal series* ($\mu \geq 1/4$), the *complementary series* ($0 < \mu < 1/4$) and *discrete series* ($\mu \leq 0$). We recall that the positive spectral values of the Casimir operator coincide with the eigenvalues of the Laplace-Beltrami operator on the (compact) hyperbolic surface, while the non-positive spectral values are given by the set of non-positive integers $\{-n^2 + n | n \in \mathbb{Z}^+\}$.

Let $\alpha \in W^r(M)$ for any $r > 3/2$ be any strictly positive function. For every Casimir parameter $\mu \in \mathbb{R}$, let

$$W_\alpha^r(H_\mu) := \{f \in L^2(M, \text{vol}_\alpha) \mid \alpha f \in W^r(H_\mu)\}.$$

By the above splitting (7), there is also a splitting

$$(8) \quad W^r(M) = \bigoplus_{\mu \in \sigma(\square)} W_\alpha^r(H_\mu)$$

For every $\mu \in \sigma(\square)$, let $\mathcal{I}_U^{-r}(H_\mu) := \mathcal{I}_U^{-r}(M) \cap W^{-r}(H_\mu)$ and let $\mathcal{I}_\alpha^{-r}(H_\mu)$ be the distributional space defined as

$$\mathcal{I}_\alpha^{-r}(H_\mu) := \{D_\alpha \mid D \in \mathcal{I}_U^{-r}(H_\mu)\}.$$

Theorem 1. *Let $\alpha \in W^r(M)$ for any $r > 3/2$. The space of U_α -invariant distributions has a splitting*

$$\mathcal{I}_\alpha^{-r}(M) = \bigoplus_{\mu \in \sigma(\square)} \mathcal{I}_\alpha^{-r}(H_\mu).$$

The subspace $\mathcal{I}_\alpha^{-r}(H_\mu)$ has dimension 2 for all irreducible sub-representations of the principal and complementary series ($\mu > 0$) and for irreducible sub-representations of the discrete series it has dimension 1 if $r > \frac{1+\sqrt{1-4\mu}}{2}$ and is trivial otherwise.

For every function $f \in W_\alpha^r(H_\mu)$, the cohomological equation $U_\alpha u = f$ has a unique solution $u \in H_\mu \subset L^2(M)$ if and only if $f \in \text{Ann}[\mathcal{I}_\alpha^{-r}(H_\mu)]$. In case a solution $u \in H_\mu$ exists then $u \in W^s(H_\mu)$ for all $s < r-1$ and the following a priori bound holds: there exists a constant $C_{s,r} > 0$, independent on $\mu \in \sigma(\square)$, such that

$$\|u\|_s \leq C_{s,r} \|\alpha f\|_r.$$

The quantitative equidistribution for time-changes, and in fact the complete asymptotics of ergodic averages, can also be derived from the corresponding results for the classical horocycle flow, derived in [5] and [2]. By change of variable there exists a function $T : M \times \mathbb{R} \rightarrow \mathbb{R}$ such that, for any function f on M ,

$$(9) \quad \begin{aligned} \int_0^\mathcal{T} f \circ h_\tau^\alpha(x) d\tau &= \int_0^{T(x,\mathcal{T})} f \circ h_{\tau(x,t)}^\alpha(x) \frac{\partial \tau}{\partial t}(x,t) dt \\ &= \int_0^{T(x,\mathcal{T})} (\alpha f) \circ h_t^U(x) dt. \end{aligned}$$

By the above formula for the constant function $f = 1$, it follows that the function $T : M \times \mathbb{R} \rightarrow \mathbb{R}$ is given by the following identity: for all $(x, \mathcal{T}) \in M \times \mathbb{R}$,

$$(10) \quad \mathcal{T} = \int_0^{T(x,\mathcal{T})} \alpha \circ h_t^U(x) dt.$$

The asymptotics of the function $T : M \times \mathbb{R} \rightarrow \mathbb{R}$ for large $\mathcal{T} \in \mathbb{R}$, uniformly with respect to $x \in M$, can be derived by the quantitative equidistribution result of M. Burger [3] (see also [5]).

Let $\mu_0 > 0$ be the smallest strictly positive eigenvalue of the Casimir operator (that, as we remarked, coincide with the smallest eigenvalue for the hyperbolic

Laplacian on the compact surface $\Gamma \setminus \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. Let $\nu_0 \in [0, 1]$ and $\epsilon_0 \in \{0, 1\}$ be the parameters defined as follows:

$$(11) \quad \nu_0 := \begin{cases} \sqrt{1-4\mu_0}, & \text{if } \mu_0 < 1/4, \\ 0, & \text{if } \mu_0 \geq 1/4; \end{cases} \quad \epsilon_0 := \begin{cases} 0, & \text{if } \mu_0 \neq 1/4, \\ 1, & \text{if } \mu_0 = 1/4. \end{cases}$$

Lemma 4. *For any $r > 3$, there exists a constant $C_r > 0$ such that the following estimate holds. Let $\alpha \in W^r(M)$. For all $(x, \mathcal{T}) \in M \times \mathbb{R}^+$,*

$$|T(x, \mathcal{T}) - \mathcal{T}| \leq C_r \|\alpha\|_r \mathcal{T}^{\frac{1+\nu_0}{2}} (1 + \log^+ \mathcal{T})^{\epsilon_0}.$$

Proof. Let us assume $\mu_0 \neq 1/4$. The argument in the case $\mu_0 = 1/4$ is similar. By the normalization condition (5), it follows from the identity (10) and from [5], Theorem 1.5, that there exists a constant $C'_r > 0$ such that, for all $(x, \mathcal{T}) \in M \times \mathbb{R}$,

$$(12) \quad |\mathcal{T} - T(x, \mathcal{T})| \leq C'_r \|\alpha\|_r \mathcal{T}^{\frac{1+\nu_0}{2}}.$$

Thus for any $v > 0$ there exists $\mathcal{T}_v > 0$ such that, for all $x \in M$ and $\mathcal{T} \geq \mathcal{T}_v$,

$$\left| \frac{\mathcal{T}}{T(x, \mathcal{T})} - 1 \right| \leq v,$$

hence $T(x, \mathcal{T}) \leq \mathcal{T}/(1-v)$. Thus by formula (12), for all $x \in M$ and for all $\mathcal{T} \geq \mathcal{T}_\sigma$,

$$|\mathcal{T} - T(x, \mathcal{T})| \leq \frac{C'_r}{(1-v)^{\frac{1+\nu_0}{2}}} \|\alpha\|_r \mathcal{T}^{\frac{1+\nu_0}{2}}.$$

The argument is thus completed. \square

By Lemma 4, the asymptotics of ergodic integrals for smooth time-changes of horocycle flows is entirely analogous and can be derived from the corresponding results for horocycle flows, proved in [5] (see also [2]).

For any $\mu > 0$, let $\nu_\mu \in \mathbb{R}$ and $\epsilon_\mu \in \{0, 1\}$ be defined as follows:

$$\nu_\mu := \begin{cases} \sqrt{1-4\mu}, & \text{if } \mu < 1/4, \\ 0, & \text{if } \mu \geq 1/4; \end{cases} \quad \epsilon_\mu := \begin{cases} 0, & \text{if } \mu \neq 1/4, \\ 1, & \text{if } \mu = 1/4. \end{cases}$$

Theorem 2. *For any $r > 3$, there exists a constant $C_r > 0$ such that the following holds. Let $\alpha \in W^r(M)$. Let H_μ be an irreducible sub-representation of the principal or complementary series ($\mu > 0$). There exists a basis $\{D_{\alpha,\mu}^+, D_{\alpha,\mu}^-\} \subset \mathcal{I}_\alpha^{-r}(H_\mu)$ such that for any function $f \in W_\alpha^r(H_\mu)$ and for all $(x, \mathcal{T}) \in M \times \mathbb{R}^+$,*

$$(13) \quad \left| \int_0^\mathcal{T} f \circ h_\tau^\alpha(x) d\tau \right| \leq C_r |D_{\alpha,\mu}^-(f)| \mathcal{T}^{\frac{1+\nu_\mu}{2}} + C_r \left(|D_{\alpha,\mu}^+(f)| \mathcal{T}^{\frac{1-\nu_\mu}{2}} (1 + \log^+ \mathcal{T})^{\epsilon_\mu} + \|f\|_r \right).$$

Let H_μ be an irreducible sub-representation of the discrete series ($\mu \leq 0$). For any function $f \in W_\alpha^r(H_\mu)$ and for all $(x, \mathcal{T}) \in M \times \mathbb{R}^+$,

$$(14) \quad \left| \int_0^\mathcal{T} f \circ h_\tau^\alpha(x) d\tau \right| \leq C_r \|f\|_r (1 + \log^+ \mathcal{T}).$$

4. QUANTITATIVE MIXING

In this section we show that the following result on quantitative mixing for smooth time-changes of horocycle flows can be derived by combining quantitative equidistribution results with B. Marcus' proof of mixing [14].

Let us denote $\|\cdot\|_X$ the graph norm of the densely defined Lie derivative operator $\mathcal{L}_X : L^2(M) \rightarrow L^2(M)$, that is, for all functions $g \in L^2(M)$ which belong to the maximal domain $\text{dom}(\mathcal{L}_X) \subset L^2(M)$ of \mathcal{L}_X ,

$$\|g\|_X := (\|g\|_0^2 + \|Xg\|_0^2)^{1/2}.$$

Theorem 3. *For any $r > 11/2$ and for any $\alpha \in W^r(M)$, there exists a constant $C_r''(\alpha) > 0$ such that the following holds. For any zero-average function $f \in W^r(M) \cap L_0^2(M, \text{vol}_\alpha)$, for any function $g \in \text{dom}(\mathcal{L}_X)$ and for any $t \geq 1$,*

$$|\langle f \circ h_t^\alpha, g \rangle_{L^2(M, \text{vol}_\alpha)}| \leq C_r(\alpha) \|f\|_r \|g\|_X t^{-\frac{1-\nu_0}{2}} (1 + \log t)^{\epsilon_0}.$$

The rest of this section is devoted to the proof of Theorem 3. The key idea of Marcus' method is to consider the push-forward under the flow of a geodesic arc.

Let $\sigma \in \mathbb{R}^+$ and $(x, t) \in M \times \mathbb{R}$. Let $\gamma_{x,t}^\sigma : [0, \sigma] \rightarrow M$ be the parametrized path defined as follows:

$$(15) \quad \gamma_{x,t}^\sigma(s) := h_t^\alpha \circ \phi_s^X(x), \quad \text{for all } s \in [0, \sigma].$$

We begin by computing the velocity of the path $\gamma_{x,t}^\sigma$ and its length. Let

$$(16) \quad v_t(x, s) := \int_0^t \left(\frac{X\alpha}{\alpha} - 1 \right) \circ h_\tau^\alpha \circ \phi_s^X(x) d\tau$$

Lemma 5. *The following identity holds for all $(x, t, s) \in M \times \mathbb{R} \times [0, \sigma]$:*

$$\frac{d\gamma_{x,t}^\sigma}{ds}(s) := v_t(x, s) U_\alpha(\gamma_{x,t}^\sigma(s)) + X(\gamma_{x,t}^\sigma(s)).$$

Proof. The velocity of the geodesic path $\{\phi_s^X(x) | s \in [0, \sigma]\}$ is given at all points by the geodesic vector field X on M , hence

$$\frac{d\gamma_{x,t}^\sigma}{ds}(s) = Dh_t^\alpha(X) \circ \phi_s^X(x).$$

The formula for the velocity of the path $\gamma_{x,t}^\sigma$ then follows from Lemma 1. \square

From the quantitative equidistribution result for the flow $\{h_t^\alpha\}$ we derive the following estimate on the velocity function.

Lemma 6. *For any $r > 4$, there exists a constant $C_r > 0$ such that the following holds. Let $\alpha \in W^r(M)$. For all $(x, s) \in M \times [0, \sigma]$ and for all $t > 0$,*

$$\left| \frac{v_t(x, s)}{t} + 1 \right| \leq C_r \|\alpha\|_r t^{-\frac{1-\nu_0}{2}} (1 + \log^+ t)^{\epsilon_0}.$$

Proof. The function $v_t(x, s)/t$ is given by an ergodic average along the trajectories of the time-change $\{h_t^\alpha\}$ (evaluated at the point $\phi_s^X(x) \in M$) of the function $X\alpha/\alpha - 1$ which has average equal to -1 with respect to the $\{h_t^\alpha\}$ -invariant volume vol_α . In fact, the latter is given by the formula $\text{vol}_\alpha = \alpha \text{vol}$, hence by the normalization condition (5) and the invariance of the volume vol under the geodesic flow,

$$\int_M \left(\frac{X\alpha}{\alpha} - 1 \right) \text{vol}_\alpha = \int_M (X\alpha - \alpha) \text{vol} = -1.$$

The result then follows from the quantitative equidistribution theorem stated above (see Theorem 2). \square

We then estimate the asymptotics (as $t \rightarrow +\infty$) of the integral

$$\int_0^\sigma (f \circ h_t^\alpha \circ \phi_s^X)(x) ds.$$

Let \hat{U}_α be the 1-form on M uniquely defined by the conditions

$$\iota_{U_\alpha} \hat{U}_\alpha = 1 \quad \text{and} \quad \iota_X \hat{U}_\alpha = \iota_V \hat{U}_\alpha = 0.$$

Lemma 7. *For any continuous function f on M , for all $\sigma > 0$ and $(x, t) \in M \times \mathbb{R}$,*

$$\int_0^\sigma (f \circ h_t^\alpha \circ \phi_s^X)(x) ds = -\frac{1}{t} \int_{\gamma_{x,t}^\sigma} f \hat{U}_\alpha + \int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X(x) \left(\frac{v_t(x,s)}{t} + 1 \right) ds.$$

Proof. By Lemma 5 and by the above definitions,

$$\int_{\gamma_{x,t}^\sigma} f \hat{U}_\alpha = \int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X(x) v_t(x,s) ds.$$

The formula follows immediately. \square

The above formula can be refined by integration by parts:

Lemma 8. *For any continuous function f on M , for all $\sigma > 0$ and $(x, t) \in M \times \mathbb{R}$,*

$$\begin{aligned} \int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X(x) ds &= -\frac{1}{t} \int_{\gamma_{x,t}^\sigma} f \hat{U}_\alpha + \left(\frac{v_t(x,\sigma)}{t} + 1 \right) \int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X(x) ds \\ &\quad - \frac{1}{t} \int_0^\sigma \frac{\partial v_t}{\partial s}(x, S) \left[\int_0^S f \circ h_t^\alpha \circ \phi_s^X(x) ds \right] dS. \end{aligned}$$

Lemma 9. *For all $s > 0$ and all $(x, t) \in M \times \mathbb{R}^+$, we have:*

$$\begin{aligned} \frac{\partial v_t}{\partial s}(x, s) &= v_t(x, s) \left(\frac{X\alpha}{\alpha} \right) \circ h_t^\alpha \circ \phi_s^X(x) \\ &\quad - \int_0^t \left[\left(\frac{X\alpha}{\alpha} - 1 \right) \left(\frac{X\alpha}{\alpha} \right) - X \left(\frac{X\alpha}{\alpha} \right) \right] \circ h_\tau^\alpha \circ \phi_s^X(x) d\tau. \end{aligned}$$

Proof. By Lemma 1 and by formula (16) it follows that

$$\frac{\partial v_t}{\partial s}(x, s) = \int_0^t [v_\tau(x, s) U_\alpha + X] \left(\frac{X\alpha}{\alpha} \right) \circ h_\tau^\alpha \circ \phi_s^X(x) d\tau.$$

By integration by parts we also have

$$\begin{aligned} \int_0^t v_\tau(x, s) U_\alpha \left(\frac{X\alpha}{\alpha} \right) \circ h_\tau^\alpha \circ \phi_s^X(x) d\tau &= \int_0^t v_\tau(x, s) \frac{d}{d\tau} \left(\frac{X\alpha}{\alpha} \right) \circ h_\tau^\alpha \circ \phi_s^X(x) d\tau \\ &= v_t(x, s) \left(\frac{X\alpha}{\alpha} \right) \circ h_t^\alpha \circ \phi_s^X(x) - \int_0^t \frac{dv_\tau}{d\tau}(x, s) \left(\frac{X\alpha}{\alpha} \right) \circ h_\tau^\alpha \circ \phi_s^X(x) d\tau \\ &= v_t(x, s) \left(\frac{X\alpha}{\alpha} \right) \circ h_t^\alpha \circ \phi_s^X(x) - \int_0^t \left[\left(\frac{X\alpha}{\alpha} - 1 \right) \left(\frac{X\alpha}{\alpha} \right) \right] \circ h_\tau^\alpha \circ \phi_s^X(x) d\tau, \end{aligned}$$

as claimed in the statement of the lemma. \square

By the Sobolev embedding theorem, we have the following estimate:

Lemma 10. *For any $r > 7/2$, there exists a constant $C_r(\alpha) > 0$ such that the following holds. Let $\alpha \in W^r(M)$. For all $(x, s) \in M \times [0, \sigma]$ and for all $t > 0$,*

$$|\frac{\partial v_t}{\partial s}(x, s)| \leq C_r(\alpha) t.$$

For any $r > 7/2$, let $C_r(\alpha) > 0$ be the constant of Lemma 10 and let

$$(17) \quad \sigma_r(\alpha) := \frac{1}{C_r(\alpha)} > 0.$$

By a bootstrap argument we derive the following bound.

Lemma 11. *For any $r > 7/2$ and for any $\sigma \in (0, \sigma_r(\alpha))$, there exist a time $t_{r,\sigma}(\alpha) > 0$ and a constant $C_{r,\sigma}(\alpha) > 0$ such that, for any continuous function f on M , for any $x \in M$ and for all $t > t_{r,\sigma}(\alpha)$,*

$$\sup_{S \in [0, \sigma]} \left| \int_0^S f \circ h_t^\alpha \circ \phi_s^X(x) ds \right| \leq C_{r,\sigma}(\alpha) \sup_{S \in [0, \sigma]} \left| \frac{1}{t} \int_{\gamma_{x,t}^S} f \hat{U}_\alpha \right|.$$

Proof. By Lemmas 8 and 10, it follows that

$$\begin{aligned} \sup_{S \in [0, \sigma]} \left| \int_0^S f \circ h_t^\alpha \circ \phi_s^X(x) ds \right| &\leq \sup_{S \in [0, \sigma]} \left| \frac{1}{t} \int_{\gamma_{x,t}^S} f \hat{U}_\alpha \right| \\ &+ \left[\max_{x \in M} \left| \frac{v_t(x, \sigma)}{t} + 1 \right| + C_r(\alpha) \sigma \right] \sup_{S \in [0, \sigma]} \left| \int_0^S f \circ h_t^\alpha \circ \phi_s^X(x) ds \right|. \end{aligned}$$

By definition $C_r(\alpha) \sigma < 1$ for any $\sigma \in (0, \sigma_r(\alpha))$, hence by Lemma 6 there exists $t_{r,\sigma}(\alpha) > 0$ such that, for all $t > t_{r,\sigma}(\alpha)$,

$$\max_{x \in M} \left| \frac{v_t(x, \sigma)}{t} + 1 \right| + C_r(\alpha) \sigma < 1.$$

Thus we conclude that the statement holds if we set

$$C_{r,\sigma}(\alpha) := [1 - \max_{x \in M} \left| \frac{v_t(x, \sigma)}{t} + 1 \right| - C_r(\alpha) \sigma]^{-1}.$$

□

We recall that the path $\gamma_{x,t}^\sigma$ is contained in a single leaf of the weak-stable foliation of the geodesic flow. By Lemma 5, its length is estimated below.

Lemma 12. *For all $r > 5/2$ and for all $\alpha \in W^r(M)$, there exists a constant $C'_r(\alpha) > 0$ such that, for all $\sigma > 0$ and for all $(x, t) \in M \times \mathbb{R}^+$,*

$$\int_{\gamma_{x,t}^\sigma} |\hat{X}| \leq C'_r(\alpha) \sigma \quad \text{and} \quad \int_{\gamma_{x,t}^\sigma} |\hat{U}| \leq C'_r(\alpha) \sigma t.$$

Proof. By Lemma 5 we have

$$\int_{\gamma_{x,t}^\sigma} |\hat{X}| = \int_0^\sigma ds \quad \text{and} \quad \int_{\gamma_{x,t}^\sigma} |\hat{U}| = \int_0^\sigma \frac{|v_t(x, s)|}{(\alpha \circ h_t^\alpha \circ \phi_s^X)(x)} ds,$$

hence the first estimate in the statement is immediate, while the second estimate follows from the uniform linear bound on $|v_t(x, s)|$ established in Lemma 6. □

From the results of [2] on the asymptotics of ergodic averages for the horocycle flow (see in particular [2], Theorem 1.3) we then derive the following bound:

Lemma 13. *Let $r > 11/2$. For any $\alpha \in W^r(M)$ and for any $\sigma > 0$, there exists a constant $C_{r,\sigma}(\alpha) > 0$ such that the following holds. For any zero-average function $f \in W^r(M) \cap L_0^2(M, \text{vol}_\alpha)$, for all $x \in M$, for all $S \in (0, \sigma]$, and for all $t > 0$,*

$$|\int_{\gamma_{x,t}^S} f \hat{U}_\alpha| \leq C_{r,\sigma}(\alpha) \|f\|_r (St)^{\frac{1+\nu_0}{2}} [1 + \log^+(St)]^{\epsilon_0}.$$

Proof. Precise bounds on the integrals

$$\int_{\gamma_{x,t}^S} f \hat{U}_\alpha = \int_{\gamma_{x,t}^S} (\alpha f) \hat{U}.$$

can be derived from [2], Theorem 1.3, if the function $\alpha f \in W^r(M)$ is supported on irreducible unitary components of the the principal and complementary series.

By Lemma 12, it follows from [2], Theorem 1.3, that there exists a constant $C_{r,\sigma}(\alpha) > 0$ such that, for all $S \in (0, \sigma]$ and for all $t > 0$,

$$|\int_{\gamma_{x,t}^S} f \hat{U}_\alpha| \leq C_{r,\sigma}(\alpha) \|f\|_r (St)^{\frac{1+\nu_0}{2}} [1 + \log^+(St)]^{\epsilon_0}.$$

In case $\alpha f \in W^r(M)$ is supported on irreducible unitary components of the discrete series, since the path $\gamma_{x,t}^\sigma$ is contained in a leaf of the weak stable foliation of the geodesic flow (tangent to the integrable distribution $\{X, U\}$) it follows from the methods of [2] that

$$|\int_{\gamma_{x,t}^S} f \hat{U}_\alpha| \leq C_{r,\sigma}(\alpha) \|f\|_r (1 + \int_{\gamma_{x,t}^S} |\hat{X}|) \log(1 + \int_{\gamma_{x,t}^S} |\hat{U}|).$$

By the above formula and by Lemma 12 the argument is completed. \square

Lemma 14. *Let $r > 11/2$. For any $\alpha \in W^r(M)$ and $\sigma \in (0, \sigma_r(\alpha))$, there exists a constant $C_{r,\sigma}(\alpha) > 0$ such that the following holds. For any zero-average function $f \in W^r(M) \cap L_0^2(M, \text{vol}_\alpha)$, for all $x \in M$, for all $S \in (0, \sigma]$, and for all $t > t_{r,\sigma}(\alpha)$,*

$$|\int_0^S (f \circ h_t^\alpha \circ \phi_s^X)(x) ds| \leq C_{r,\sigma}(\alpha) \|f\|_r (St)^{\frac{1+\nu_0}{2}} [1 + \log^+(St)]^{\epsilon_0} / t.$$

Proof. By the bounds proved in Lemma 6 and Lemma 11, the statement follows immediately from the above Lemma 13. \square

By the invariance of the the standard volume under the geodesic flow and by integration by parts we derive the following formula.

Lemma 15. *Let $\sigma > 0$. For all $f \in L^2(M)$ and for all $g \in L^2(M)$ which belong to the maximal domain $\text{dom}(\mathcal{L}_X) \subset L^2(M)$ of the densely defined Lie derivative operator $\mathcal{L}_X : L^2(M) \rightarrow L^2(M)$, for all $t \in \mathbb{R}$,*

$$\begin{aligned} \langle f \circ h_t^\alpha, g \rangle &= \frac{1}{\sigma} \langle \int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X ds, g \circ \phi_\sigma^X \rangle \\ &\quad - \frac{1}{\sigma} \int_0^\sigma \langle \int_0^S f \circ h_t^\alpha \circ \phi_s^X ds, \mathcal{L}_X g \circ \phi_S^X \rangle dS \end{aligned}$$

It is immediate to derive from Lemma 14 and from Lemma 15 estimates on the decay of correlations for sufficiently smooth functions in Theorem 3.

Proof of Theorem 3. By Lemma 14 and Lemma 15, there exists a constant $C'_r(\alpha) > 0$ such that

$$(18) \quad | \langle f \circ h_t^\alpha, g \rangle | \leq C'_r(\alpha) \|f\|_r \|g\|_X t^{-\frac{1-\nu_0}{2}} (1 + \log t)^{\epsilon_0}.$$

Finally, by taking into account that, for all $f, g \in W^r(M)$ and all $t \geq 0$,

$$\langle f \circ h_t^\alpha, g \rangle_{L^2(M, \text{vol}_\alpha)} = \langle f \circ h_t^\alpha, \alpha g \rangle,$$

the theorem follows from the estimate in formula (18). In fact, by the Sobolev embedding theorem, for all $g \in \text{dom}(\mathcal{L}_X)$,

$$\|\alpha g\|_X \leq 2\|\alpha\|_r \|g\|_X.$$

□

5. TWISTED ERGODIC INTEGRALS: SQUARE-MEAN ESTIMATES

We prove below L^2 bounds on twisted ergodic integrals of smooth functions, that is, on integrals of the form

$$\int_0^\mathcal{T} w(t) f \circ h_t^\alpha(x) dt$$

for any function $w \in L^\infty(\mathbb{R}, \mathbb{C})$ and for any sufficiently smooth function f on M . In the next sections we will derive from these bounds estimates on spectral measures of smooth time-changes of the horocycle flow. The relevant twisted integrals are those with twist function equal to an exponential function which are related to the Fourier transforms of spectral measures.

We remark that the question on optimal *uniform* bounds for twisted ergodic integrals (with an exponential twist) is open even for the classical horocycle flow. Non-optimal, polynomial bounds can be derived from estimates on the rate of equidistribution and on the rate of mixing by an argument due to A. Venkatesh [23]. Such uniform bounds are closely related to estimates on the rate of equidistribution of time- \mathcal{T} maps of horocycle flows.

For our main results on spectral measures of time-changes, uniform bounds on twisted ergodic integrals are not needed. In fact, the key step is to prove square-mean estimates of the type below.

Lemma 16. *Let $r > 11/2$ and let $\alpha \in W^r(M)$. There exists $\sigma_r(\alpha) > 0$ such that for all $\sigma \in (0, \sigma_r(\alpha))$ the following holds. There exists a constant $C_{r,\sigma}(\alpha) > 0$ such that for any bounded weight function $w \in L^\infty(\mathbb{R}^+, \mathbb{C})$, for any continuous function $f \in \text{dom}(X) \subset L^2(M, \text{vol}_\alpha)$ and for all $\mathcal{T} > 0$,*

$$\begin{aligned} \left\| \int_0^\mathcal{T} w(t) f \circ h_t^\alpha dt \right\|_{L^2(M, \text{vol}_\alpha)} &\leq C_{r,\sigma}(\alpha) |w|_\infty \|f\|_X^{1/2} \\ &\times \left[|f|_\infty^{1/2} \mathcal{T} + \int_0^\mathcal{T} \int_0^t \left\| \sup_{S \in [0, \sigma]} \left| \int_{\gamma_{x,\tau}^S} f \hat{U}_\alpha \right| \right\|_0 \frac{d\tau dt}{\tau} \right]^{1/2}. \end{aligned}$$

Proof. By the invariance of the volume form under the reparametrized horocycle flow and by change of variables we have that

$$\left\| \int_0^\mathcal{T} w(t) f \circ h_t^\alpha dt \right\|_{L^2(M, \text{vol}_\alpha)}^2 = 2\text{Re} \left\langle \int_0^\mathcal{T} \int_0^t w(t) \overline{w(t-\tau)} f \circ h_\tau^\alpha d\tau dt, \alpha f \right\rangle.$$

For every fixed $t \in \mathbb{R}$, let $w_t \in C^0(\mathbb{R}, \mathbb{C})$ be the bounded weight function defined as

$$w_t(\tau) := \overline{w(t - \tau)}, \quad \text{for all } \tau \in \mathbb{R}.$$

By the formula of Lemma 15 we have that

$$\begin{aligned} & \left\langle \int_0^\mathcal{T} \int_0^t w(t) w_t(\tau) f \circ h_\tau^\alpha d\tau dt, \alpha f \right\rangle \\ &= \frac{1}{\sigma} \int_0^\mathcal{T} \int_0^t w(t) w_t(\tau) \left\langle \int_0^\sigma f \circ h_\tau^\alpha \circ \phi_s^X ds, (\alpha f) \circ \phi_\sigma^X \right\rangle d\tau dt \\ &= \frac{1}{\sigma} \int_0^\mathcal{T} \int_0^t w(t) w_t(\tau) \int_0^\sigma \left\langle f \circ h_\tau^\alpha \circ \phi_s^X ds, \mathcal{L}_X(\alpha f) \circ \phi_s^X \right\rangle dS d\tau dt. \end{aligned}$$

The statement of the lemma then follows from Lemma 11 and from the estimate

$$\left\| \int_0^\mathcal{T} \int_0^{t_{r,\sigma}(\alpha)} \int_0^S w(t) w_t(\tau) f \circ h_\tau^\alpha \circ \phi_s^X ds d\tau dt \right\|_0 \leq |w|_\infty^2 |f|_\infty t_{r,\sigma}(\alpha) S \mathcal{T}.$$

□

Theorem 4. *Let $r > 11/2$. For any $\alpha \in W^r(M)$, there exists a constant $C_r(\alpha) > 0$ such that the following holds. For any bounded weight function $w \in L^\infty(\mathbb{R}^+, \mathbb{C})$, for any zero-average function $f \in W^r(M) \cap L_0^2(M, \text{vol}_\alpha)$ and for all $\mathcal{T} > 0$,*

$$\left\| \int_0^\mathcal{T} w(t) f \circ h_t^\alpha dt \right\|_{L^2(M, \text{vol}_\alpha)} \leq C_r(\alpha) |w|_\infty \|f\|_r \mathcal{T}^{\frac{3+\nu_0}{4}} [1 + \log^+ \mathcal{T}]^{\frac{\epsilon_0}{2}}.$$

Proof. By the equidistribution estimates proved in Lemma 13, for any $\sigma > 0$ there exists a constant $C_{r,\sigma}(\alpha) > 0$ such that for all $f \in W^r(M) \cap L_0^2(M, \text{vol}_\alpha)$ and for all $x \in M$ and all $\tau > 0$,

$$\sup_{S \in [0, \sigma]} \left| \int_{\gamma_{x,\tau}^S} f \hat{U}_\alpha \right| \leq C_{r,\sigma}(\alpha) \|f\|_r \tau^{\frac{1+\nu_0}{2}} [1 + \log^+ \tau]^{\epsilon_0}.$$

The statement of the theorem then follows from Lemma 16 by integration. □

We remark that more refined estimates can be proved for functions supported on finite codimensional subspaces orthogonal to irreducible components of the complementary series and for coboundaries. From the estimates for coboundaries, which will be fully carried out below, we will deduce that all smooth time-changes of the horocycle flow have absolutely continuous spectrum.

6. SPECTRAL THEORY

In this final section we state and prove spectral results for smooth time changes of horocycle flows. In Section 6.1 we first show a local estimate on spectral measures of smooth functions (see Theorem 5). Exploiting the L_2 -bounds established in the previous Section 5, in Section 6.2 we prove the absolute continuity of the spectrum (with countable multiplicity) for all smooth time-changes of the classical horocycle flow (Theorem 6). Finally in Section 6.3 we show that the maximal spectral type is always equivalent to Lebesgue (Theorem 7).

6.1. Local estimates. Let μ_f denote the spectral measures of a function $f \in L^2(M, \text{vol}_\alpha)$. We recall that μ_f is a complex measure on the real line. The main result derived in this section is the following.

Theorem 5. *Let $r > 11/2$ and let $\alpha \in W^r(M)$. There exists a constant $C_r(\alpha) > 0$ such that, for any function $u \in W^{r+1}(M)$ and for any $\xi \in \mathbb{R} \setminus \{0\}$,*

$$|\mu_u(\xi - \delta, \xi + \delta)| \leq C_r(\alpha) \|u\|_{r+1} \frac{\delta |\log \delta|}{\xi^2}, \quad \text{for all } \delta \in (0, |\xi|/2),$$

hence the measure μ_u has local dimension 1 at all points $\xi \in \mathbb{R} \setminus \{0\}$, that is,

$$\lim_{\delta \rightarrow 0^+} \frac{\log \mu_u(\xi - \delta, \xi + \delta)}{\log \delta} = 1.$$

The above result implies that the local dimension of spectral measures of smooth functions is everywhere equal to 1, but it is off by a logarithmic term from the sharpest possible bound, which would imply that spectral measures of sufficiently smooth functions are absolutely continuous with bounded densities. In the following section (§6.2) we nevertheless show how one can derive absolute continuity from the mean-square bounds for ergodic integrals of coboundaries.

We first estimate twisted ergodic integrals of coboundaries. Let us assume that $f \in L^2(M)$ is a smooth coboundary for the time-change $\{h_t^\alpha\}$ on M , that is, there exists a function $u \in W^r(M)$ such that

$$f = U_\alpha u.$$

Lemma 17. *There exists a constant $C > 0$ such that for all $\sigma > 0$, for all continuous coboundaries $f = U_\alpha u$ with transfer function $u \in L^\infty(M)$ such that $Xu \in L^\infty(M)$, for all $x \in M$ and all $t > 0$,*

$$\sup_{S \in [0, \sigma]} \left| \int_{\gamma_{x,t}^S} f \hat{U}_\alpha \right| \leq C \max\{1, \sigma\} \max\{\|u\|_{L^\infty(M)}, \|Xu\|_{L^\infty(M)}\}.$$

Proof. By the definition of the path $\gamma_{x,t}^\sigma$ in formula (15) and by Lemma 5 we have

$$\begin{aligned} \int_{\gamma_{x,t}^S} f \hat{U}_\alpha &= \int_{\gamma_{x,t}^S} du - \int_{\gamma_{x,t}^S} Xu \hat{X} \\ (19) \quad &= u \circ h_t^\alpha \circ \phi_S^X(x) - u \circ h_t^\alpha(x) - \int_0^S Xu \circ h_t^\alpha \circ \phi_s^X(x) ds, \end{aligned}$$

hence the statement of the lemma follows. \square

Corollary 1. *Let $r > 11/2$ and let $\alpha \in W^r(M)$. There exists a constant $C_r(\alpha) > 0$ such that for any bounded weight function $w \in L^\infty(\mathbb{R}^+, \mathbb{C})$, for all coboundaries $f = U_\alpha u$ with a transfer function $u \in W^{r+1}(M)$ and for all $\mathcal{T} > 0$,*

$$\left\| \int_0^\mathcal{T} w(t) f \circ h_t^\alpha dt \right\|_{L^2(M, \text{vol}_\alpha)} \leq C_r(\alpha) |w|_\infty \|u\|_{r+1} \mathcal{T}^{1/2} (1 + \log^+ \mathcal{T})^{1/2}.$$

Proof. By Sobolev embedding theorem if $u \in W^{r+1}(M)$ then $u, Xu \in L^\infty(M)$ and the following estimate holds: there exists a constant $C_r > 0$ such that

$$\max\{\|u\|_{L^\infty(M)}, \|Xu\|_{L^\infty(M)}\} \leq C_r \|u\|_{r+1}.$$

The statement then follows by integration from Lemma 16 and Lemma 17. \square

Proof of Theorem 5. By the spectral theorem, for any function $f \in L^2(M, \text{vol}_\alpha)$ and any $\xi \in \mathbb{R}$,

$$(20) \quad \begin{aligned} \left\| \frac{e^{i(\xi+\eta)\mathcal{T}} - 1}{i(\xi + \eta)} \right\|_{L_\eta^2(\mathbb{R}, d\mu_f)}^2 &= \left\| \int_0^\mathcal{T} e^{i(\xi+\eta)t} dt \right\|_{L_\eta^2(\mathbb{R}, d\mu_f)}^2 \\ &= \left\| \int_0^\mathcal{T} e^{i\xi t} f \circ h_t^\alpha dt \right\|_0^2. \end{aligned}$$

By a simple computation, there exists a constant $C > 0$ such that

$$(21) \quad \begin{aligned} \mu_f\left(\xi - \frac{1}{\mathcal{T}}, \xi + \frac{1}{\mathcal{T}}\right) &\leq C \int_{-\frac{1}{\mathcal{T}}}^{\frac{1}{\mathcal{T}}} \left| \frac{e^{i(\xi+\eta)\mathcal{T}} - 1}{i(\xi + \eta)\mathcal{T}} \right|^2 d\mu_f(\eta) \\ &\leq \frac{C}{\mathcal{T}^2} \left\| \frac{e^{i(\xi+\eta)\mathcal{T}} - 1}{i(\xi + \eta)} \right\|_{L_\eta^2(\mathbb{R}, d\mu_f)}^2. \end{aligned}$$

Let $u \in \text{dom}(U_\alpha) \subset L^2(M, \text{vol}_\alpha)$ and let $f := U_\alpha u \in L^2(M, \text{vol}_\alpha)$. By the spectral theorem we have that

$$(22) \quad d\mu_f(\xi) = \xi^2 d\mu_u(\xi), \quad \text{for all } \xi \in \mathbb{R},$$

hence there exists a constant $C' > 0$ such that, for all $\xi \in \mathbb{R} \setminus \{0\}$,

$$(23) \quad \mu_u(\xi - \delta, \xi + \delta) \leq C' \frac{\mu_f(\xi - \delta, \xi + \delta)}{\xi^2}, \quad \text{for all } \delta \in (0, |\xi|/2).$$

By formulas (20) and (21) and by Corollary 1, it follows that there exists a constant $C_r(\alpha) > 0$ such that, for all functions $u \in W^{r+1}(M)$ and for all $\mathcal{T} > 0$,

$$(24) \quad \mu_f\left(\xi - \frac{1}{\mathcal{T}}, \xi + \frac{1}{\mathcal{T}}\right) \leq C_r(\alpha) \|u\|_{r+1} \frac{(1 + \log^+ \mathcal{T})}{\mathcal{T}}.$$

The statement of the theorem can readily be derived from formulas (23) and (24). \square

6.2. Absolute continuity. We show in this section that the spectral measure μ_f of any function $f \in L^2(M, \text{vol}_\alpha)$ is absolutely continuous with respect to the Lebesgue measure on the real line and hence that any smooth time-change of the horocycle flow has absolutely continuous spectrum.

Theorem 6. *Let $r > 11/2$ and let $\alpha \in W^r(M)$. The time-change $\{h_t^\alpha\}$ of the (stable) horocycle flow $\{h_t^U\}$ with infinitesimal generator $U_\alpha := U/\alpha$ has purely absolutely continuous spectrum.*

The Theorem is derived below from the following estimate on decay of correlations of coboundaries.

Lemma 18. *Let $r > 11/2$ and let $\alpha \in W^r(M)$. There exist constants $C_r(\alpha) > 0$ and $t_r(\alpha) > 0$ such that the following holds. For any continuous coboundary $f = U_\alpha u$ with a transfer function $u \in L^\infty(M)$ such that $Xu \in L^\infty(M)$ and for any $g \in \text{dom}(X)$, for all $t > t_r(\alpha)$,*

$$|\langle f \circ h_t^\alpha, g \rangle| \leq C_r(\alpha) \|u\|_{r+1} \|g\|_X (1/t).$$

Proof. The statement follows readily from Lemma 11, Lemma 15 and Lemma 17. \square

Proof of Theorem 6. Since coboundaries with smooth transfer functions are dense in $L^2(M, \text{vol}_\alpha)$, it is enough to prove that their spectral measures are absolutely continuous. In fact, we will prove that for any coboundary $f = U_\alpha u$ with transfer function $u \in W^{r+1}(M)$ its spectral measure is absolutely continuous with square-integrable density. Let $\hat{\mu}_f$ be the Fourier transform of the spectral measure μ_f , which is the bounded function defined as follows:

$$\hat{\mu}_f(t) := \int_{\mathbb{R}} e^{i\xi t} d\mu_f(\xi), \quad \text{for all } t \in \mathbb{R}.$$

By definition of the spectral measures, for all $t \in \mathbb{R}$ we have the following identity

$$\hat{\mu}_f(t) = \langle f \circ h_t^\alpha, f \rangle_{L^2(M, \text{vol}_\alpha)} = \langle f \circ h_t^\alpha, \alpha f \rangle.$$

By Lemma 18 we therefore have the following estimate: for any $t > t_r(\alpha)$,

$$(25) \quad |\hat{\mu}_f(t)| = |\langle f \circ h_t^\alpha, \alpha f \rangle| \leq C_r(\alpha) \|u\|_{r+1} \|\alpha f\|_X (1/t).$$

Since $\hat{\mu}_f$ is a bounded function and it is symmetric, that is, for all $t \in \mathbb{R}$,

$$\hat{\mu}_f(t) = \langle f \circ h_t^\alpha, f \rangle_{L^2(M, \text{vol}_\alpha)} = \langle f, f \circ h_{-t}^\alpha \rangle_{L^2(M, \text{vol}_\alpha)} = \overline{\hat{\mu}_f(-t)},$$

and since the function $1/t \in L^2((t_r(\alpha), +\infty), dt)$, we have proved that the Fourier transform $\hat{\mu}_f \in L^2(\mathbb{R}, dt)$, which readily implies that the spectral measure μ_f is absolutely continuous with square-integrable Radon-Nikodym derivative. In fact, there exists a constant $C_r''(\alpha) > 0$ such that the following estimate holds:

$$\left\| \frac{D\mu_f}{D\xi} \right\|_{L^2(\mathbb{R}, d\xi)} = \|\hat{\mu}_f\|_{L^2(\mathbb{R}, dt)} \leq C_r''(\alpha) \|u\|_{r+1} \|f\|_X.$$

The proof of the theorem is complete. \square

6.3. Maximal spectral type. Let us now prove that the *maximal spectral type* of any smooth time-change of the classical horocycle flow is equivalent to Lebesgue.

Lemma 19. *For any $\sigma \in (0, \sigma_r(\alpha))$ there exists a constant $C'_{r,\sigma}(\alpha) > 0$ such that, for all $x \in M$, for all $t > t_r(\alpha)$ and for all functions $u \in C^1(M)$, we have*

$$\left| \int_0^\sigma U_\alpha u \circ h_t^\alpha \circ \phi_s^X(x) ds \right| \leq C'_{r,\sigma}(\alpha) \max\{\|u\|_{L^\infty(M)}, \|Xu\|_{L^\infty(M)}\} (1/t).$$

Proof. The statement follows from Lemma 11 and Lemma 17. \square

Lemma 20. *Let $r > 11/2$ and let $\alpha \in W^r(M)$. Assume that the maximal spectral type of the time-change $\{h_t^\alpha\}$ is not Lebesgue. There exists a smooth non-zero function $w \in L^2(\mathbb{R}, dt)$ such that for all $x \in M$, for all $\sigma \in (0, \sigma_r(\alpha))$ and for all functions $u \in C^1(M)$ the following holds:*

$$\int_{\mathbb{R}} w(t) \int_0^\sigma U_\alpha u \circ h_t^\alpha \circ \phi_s^X(x) ds dt = 0$$

Proof. If the maximal spectral type is not Lebesgue, then the Lebesgue measure is not absolutely continuous with respect to the maximal spectral measure. Thus, there exists a compact set $A \subset \mathbb{R}$ such that A has measure zero with respect to the maximal spectral measure of the flow $\{h_t^\alpha\}$, hence with respect to all its spectral measures, but A has strictly positive Lebesgue measure.

Let $w \in L^2(\mathbb{R})$ be the complex conjugate of the Fourier transform of the characteristic function χ_A of the set $A \subset \mathbb{R}$. For any pair of functions $f, g \in L^2(M)$

let $\mu_{f,g}$ denote the joint spectral measure. By Theorem 6 the measure $\mu_{f,g}$ is absolutely continuous with respect to Lebesgue. Whenever $\mu_{f,g}$ has square-integrable density we have

$$(26) \quad \int_{\mathbb{R}} w(t) \langle f \circ h_t^\alpha, g \rangle_{L^2(M, \text{vol}_\alpha)} dt = \int_{\mathbb{R}} \chi_A(\xi) d\mu_{f,g}(\xi) = 0.$$

It follows from Lemma 18 that, whenever $f = U_\alpha u$ is a coboundary with transfer function $u \in C^1(M)$, then the Fourier transform of the spectral measure $\mu_{f,g}$, hence its density, is square-integrable, so that the identity in formula (26) holds.

From formula (26) by translation under the geodesic flow and by integration we derive that, for any $\sigma > 0$ and for any function $g \in W^r(M) \subset \text{dom}(X)$,

$$(27) \quad \int_0^\sigma \int_{\mathbb{R}} w(t) \langle U_\alpha u \circ h_t^\alpha \circ \phi_s^X, g \rangle_{L^2(M, \text{vol}_\alpha)} dt ds = 0.$$

By Lemma 18 the double integral in formula (27) is absolutely convergent, hence

$$(28) \quad \langle \int_{\mathbb{R}} w(t) \int_0^\sigma U_\alpha u \circ h_t^\alpha \circ \phi_s^X(\cdot) ds dt, g \rangle_{L^2(M, \text{vol}_\alpha)} = 0.$$

It follows from Lemma 19 that the function

$$\int_{\mathbb{R}} w(t) \int_0^\sigma U_\alpha u \circ h_t^\alpha \circ \phi_s^X(\cdot) ds dt$$

is bounded on M , hence it vanishes identically by formula (28) and by density of the subspace $W^r(M) \subset L^2(M)$. \square

Lemma 21. *Let $w \in L^2(\mathbb{R}, dt)$ be a smooth function. Assume that for some $x \in M$, for some $\sigma \in (0, \sigma_r(\alpha))$ and for all functions $u \in C^1(M)$,*

$$\int_{\mathbb{R}} w(t) \int_0^\sigma U_\alpha u \circ h_t^\alpha \circ \phi_s^X(x) ds dt = 0.$$

It follows that w vanishes identically.

Proof. Let us fix $x \in M$ and $\sigma > 0$. For any given $T > 0$ and $\rho > 0$, let $E_{\rho, \sigma}^T$ be the flow-box for the time-change $\{h_t^\alpha\}$ defined as follows:

$$(29) \quad E_{\rho, \sigma}^T(r, s, t) = (h_t^\alpha \circ \phi_s^X \circ h_r^V)(x), \quad \text{for all } (r, s, t) \in (-\rho, \rho) \times (-\sigma, \sigma) \times (-T, T).$$

Since the horocycle flow has no periodic orbits, it never returns to any given geodesic segment, hence for any $\sigma > 0$ and any $T > 0$ there exists $\rho > 0$ such that $E_{\rho, \sigma}^T$ is an embedding. For any $\chi \in C^\infty(-1, 1)$ and any $\psi \in C_0^\infty(-T, T)$, let

$$\tilde{u}(r, s, t) := \chi\left(\frac{r}{\rho}\right) \chi\left(\frac{s}{\sigma}\right) \psi(t), \quad \text{for all } (r, s, t) \in (-\rho, \rho) \times (-\sigma, \sigma) \times (-T, T).$$

Let $u \in C^\infty(M)$ be the function defined as $u = 0$ on $M \setminus \text{Im}(E_{\rho, \sigma}^T)$ and as

$$(30) \quad (u \circ E_{\rho, \sigma}^T)(r, s, t) := \tilde{u}(r, s, t), \quad \text{for all } (r, s, t) \in (-\rho, \rho) \times (-\sigma, \sigma) \times (-T, T),$$

on $\text{Im}(E_{\rho,\sigma}^T)$. We claim that the following formulas hold:

$$(31) \quad \begin{aligned} (U_\alpha u) \circ E_{\rho,\sigma}^T(r, s, t) &:= \chi\left(\frac{r}{\rho}\right) \chi\left(\frac{s}{\sigma}\right) \frac{d\psi}{dt}(t), \\ (Xu) \circ E_{\rho,\sigma}^T(r, s, t) &:= \sigma \chi\left(\frac{r}{\rho}\right) \frac{d\chi}{ds}\left(\frac{s}{\sigma}\right) \psi(t) \\ &\quad - v_t(h_r^V(x), s) \chi\left(\frac{r}{\rho}\right) \chi\left(\frac{s}{\sigma}\right) \frac{d\psi}{dt}(t). \end{aligned}$$

In fact, the above formulas are immediate consequence of the identities below. Let $\mathcal{T}_{r,s} : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the unique solution of the parametric Cauchy problem

$$(32) \quad \begin{cases} \frac{\partial \mathcal{T}_{r,s}}{\partial S}(t, S) &= -v_{\mathcal{T}_{r,s}(t,S)}(h_r^V(x), s + S), \\ \mathcal{T}_{r,s}(t, 0) &= t. \end{cases}$$

For any $(r, s, t) \in (-\rho, \rho) \times (-\sigma, \sigma) \times (-T, T)$ there exists $(\tau_0, S_0) \in \mathbb{R}^+$ such that, for all $(\tau, S) \in (-\tau_0, \tau_0) \times (-S_0, S_0)$, the following holds:

$$(33) \quad \begin{aligned} h_r^\alpha \circ E_{\rho,\sigma}^T(r, s, t) &= E_{\rho,\sigma}^T(r, s, t + \tau), \\ \phi_S^X \circ E_{\rho,\sigma}^T(r, s, t) &= E_{\rho,\sigma}^T(r, s + S, \mathcal{T}_{r,s}(t, S)). \end{aligned}$$

The first of the above formulas is an immediate consequence of the definition (29) of the flow-box map. The second formula follows from the commutation relation:

$$(34) \quad \phi_S^X \circ h_t^\alpha \circ \phi_s^X \circ h_r^V(x) = h_{\mathcal{T}_{r,s}(t,S)} \circ \phi_{s+S}^X \circ h_r^V(x), \quad \text{for all } (t, s, S) \in \mathbb{R}^3.$$

Let us prove the above commutation identity. For $S = 0$ it holds for all $(r, s, t) \in \mathbb{R}^3$. For all $(r, s) \in \mathbb{R}^2$ let $x_{r,s} := (\phi_s^X \circ h_r^V)(x)$. By Lemma 1, by differentiation of equation (34) with respect to $S \in \mathbb{R}$, we find

$$\begin{aligned} (X \circ \phi_S^X \circ h_t^\alpha)(x_{r,s}) &= \frac{\partial \mathcal{T}_{r,s}}{\partial S}(t, S) (U_\alpha \circ h_{\mathcal{T}_{r,s}(t,S)}^\alpha \circ \phi_S^X)(x_{r,s}) \\ &\quad + v_{\mathcal{T}_{r,s}(t,S)}(x_{r,s}, S) (U_\alpha \circ h_{\mathcal{T}_{r,s}(t,S)}^\alpha \circ \phi_S^X)(x_{r,s}) \\ &\quad + (X \circ h_{\mathcal{T}_{r,s}(t,S)}^\alpha \circ \phi_S^X)(x_{r,s}). \end{aligned}$$

Since the above equation holds by the definition of the function $\mathcal{T}_{r,s}$ in formula (32), the commutation relation (34) holds as well. We have thus proved the flow-box identities (33) from which the differentiation formulas (31) follow immediately.

For any $\rho, \sigma > 0$, let $T_{\rho,\sigma} > 0$ be defined as follows:

$$(35) \quad T_{\rho,\sigma} := \min\{|t| > T \mid \cup_{s \in [-\sigma, \sigma]} (h_t^\alpha \circ \phi_s^X)(x) \cap \text{Im}(E_{\rho,\sigma}^T) \neq \emptyset\}.$$

Since the horocycle flow never returns to any given geodesic segment, for every fixed $\sigma > 0$, the following holds:

$$(36) \quad \lim_{\rho \rightarrow 0^+} T_{\rho,\sigma} = +\infty.$$

By assumption and by formula (31) we have

$$(37) \quad \begin{aligned} &\chi(0) \left(\int_0^\sigma \chi\left(\frac{s}{\sigma}\right) ds \right) \left(\int_{-T}^T w(t) \frac{d\psi}{dt}(t) dt \right) \\ &\quad + \int_{\mathbb{R} \setminus [-T_{\rho,\sigma}, T_{\rho,\sigma}]} w(t) \int_0^\sigma U_\alpha u \circ h_t^\alpha \circ \phi_s^X(x) ds dt = 0. \end{aligned}$$

We claim that the following holds: for any fixed $\sigma \in (0, \sigma_r(\alpha))$ and $T > 0$,

$$(38) \quad \lim_{\rho \rightarrow 0^+} \int_{\mathbb{R} \setminus [-T_{\rho, \sigma}, T_{\rho, \sigma}]} w(t) \int_0^\sigma U_\alpha u \circ h_t^\alpha \circ \phi_s^X(x) ds dt = 0.$$

Since the function $u \in C^\infty(M)$, by Lemma 19, combined with a trivial estimate for $0 \leq t \leq t_r(\alpha)$, there exists a constant $C'_{r, \sigma}(\alpha) > 0$ such that, for all $x \in M$ and for all $\sigma \in (0, \sigma_r(\alpha))$, we have

$$(39) \quad \begin{aligned} & \left\| \int_0^\sigma U_\alpha u \circ h_t^\alpha \circ \phi_s^X(x) ds \right\|_{L^2(\mathbb{R}, dt)} \\ & \leq C'_{r, \sigma}(\alpha) \max\{\|u\|_{L^\infty(M)}, \|Xu\|_{L^\infty(M)}, \|U_\alpha u\|_{L^\infty(M)}\}. \end{aligned}$$

By the definition of the function $u \in C^\infty(M)$ (see formula (30)), by the trivial estimate $|v_t| \leq C_\alpha t \leq C_\alpha T$ and by the estimates in formula (31) we also have

$$(40) \quad \begin{aligned} & \max\{\|u\|_{L^\infty(M)}, \|Xu\|_{L^\infty(M)}, \|U_\alpha u\|_{L^\infty(M)}\} \leq C'_{r, \sigma}(\alpha) \max\{1, T\} \\ & \times \max^2\{\|\chi\|_{L^\infty(\mathbb{R})}, \|\chi'\|_{L^\infty(\mathbb{R})}\} \max\{\|\psi\|_{L^\infty(\mathbb{R})}, \|\psi'\|_{L^\infty(\mathbb{R})}\}. \end{aligned}$$

In particular the above bound is uniform with respect to $\rho > 0$. Thus the limit in formula (38) follows from the uniform L^2 bound given by formulas (39) and (40).

Since formulas (37) and (38) hold for all functions $\chi \in C_0^\infty(-1, 1)$, for all $T > 0$ and for all functions $\psi \in C_0^\infty(-T, T)$, it follows that

$$\int_{\mathbb{R}} w(t) \frac{d\psi}{dt}(t) dt = 0, \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}),$$

hence the function $w \in L^2(\mathbb{R}, dt)$ is a constant (necessarily equal to zero). \square

By Lemma 20 and Lemma 21 we derive our conclusive spectral result.

Theorem 7. *Let $r > 11/2$ and let $\alpha \in W^r(M)$. The maximal spectral type of the time-change $\{h_t^\alpha\}$ of the (stable) horocycle flow $\{h_t^U\}$ with infinitesimal generator $U_\alpha := U/\alpha$ is equivalent to Lebesgue.*

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REFERENCES

- [1] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. (2) **48**, 1947, 568–640.
- [2] A. Bufetov & G. Forni, *Limit Theorems for Horocycle Flows*, preprint 2011, 1–52 (arXiv:1104.4502v1).
- [3] M. Burger, *Horocycle flow on geometrically finite surfaces*, Duke Math. J. **61**, 1990, 779–803.
- [4] S. G. Dani, *Invariant measures and minimal sets of horospherical flows*, Invent. Math. **64**, 1981, no. 2, 357–385.
- [5] L. Flaminio & G. Forni, *Invariant distributions and time averages for horocycle flows*, Duke Math. J. **119**, 2003, no. 3, 465–526.

- [6] H. Furstenberg, *The unique ergodicity of the horocycle flow*, in Recent Advances in Topological Dynamics (New Haven, Conn., 1972), Lecture Notes in Math. **318**, Springer, Berlin, 1973, 95–115.
- [7] I. M. Gelfand & S. V. Fomin, *Unitary representations of Lie groups and geodesic flows on surfaces of constant negative curvature* (in Russian), Dokl. Akad. Nauk SSSR **76**, 1951, 771–774.
- [8] I. M. Gelfand & M. Neumark, *Unitary representations of the Lorentz group*, Acad. Sci. USSR. J. Phys. **10**, 1946, 93–94.
- [9] B. M. Gurevich, *The entropy of horocycle flows* (in Russian), Dokl. Akad. Nauk SSSR **136**, 1961, 768–770.
- [10] G. A. Hedlund, *Fuchsian groups and transitive horocycles*, Duke Math. J. **2**, 1936, 530–542.
- [11] D. A. Hejhal, *On the uniform equidistribution of long closed horocycles*. Loo-Keng Hua: a great mathematician of the twentieth century. Asian J. Math. **4**, 2000, no. 4, 839–853.
- [12] A. Katok & J.-P. Thouvenot, *Spectral Properties and Combinatorial Constructions in Ergodic Theory*, Handbook of Dynamical Systems, Vol. 1B (B. Hasselblatt and A. Katok editors), Elsevier, 2006, 649–743.
- [13] A. G. Kushnirenko, *Spectral properties of some dynamical systems with polynomial divergence of orbits*, Vestnik Moskovskogo Universiteta. Matematika, **29**, No. 1, 1974, 101–108.
- [14] B. Marcus, *Ergodic properties of horocycle flows for surfaces of negative curvature*, Ann. of Math. (2) **105**, 1977, 81–105.
- [15] C. C. Moore, *Exponential decay of correlation coefficients for geodesic flows*, Group Representations Ergodic Theory, Operator Algebras, and Mathematical Physics (Berlin, Heidelberg, New York) (C. C. Moore, ed.), Mathematical Science Research Institute Publications, vol. **6**, Springer-Verlag, 1987, 163–181.
- [16] O. S. Parasyuk, *Flows of horocycles on surfaces of constant negative curvature* (in Russian), Uspekhi Mat. Nauk **8**, no. 3, 1953, 125–126.
- [17] M. Ratner, *Factors of horocycle flows*, Ergodic Theory Dynam. Systems **2**, 1982, 465–489.
- [18] ———, *Rigidity of horocycle flows*, Ann. of Math. (2) **115**, 1982, 597–614.
- [19] ———, *Horocycle flows, joinings and rigidity of products*, Ann. of Math. (2) **118**, 1983, 277–313.
- [20] ———, *The rate of mixing for geodesic and horocycle flows*, Ergodic Theory Dynam. Systems **7**, 1987, 267–288.
- [21] P. Sarnak, *Asymptotic behavior of periodic orbits of the horocycle flow and Eisenstein series*, Comm. Pure Appl. Math. **34**, 1981, 719–739.
- [22] A. Strömbergsson, *On the uniform equidistribution of long closed horocycles*. Duke Math. J. **123**, 2004, no. 3, 507–547.
- [23] A. Venkatesh, *Sparse equidistribution problems, period bounds and subconvexity*, Ann. of Math. (2) **172**, 2010, 989–1094.
- [24] D. Zagier, *Eisenstein series and the Riemann zeta function*, in Automorphic Forms, Representation Theory and Arithmetic (Bombay, 1979), Tata Inst. Fund. Res. Studies in Math. **10**, Tata Inst. Fund. Res., Bombay, 1981, 275–301.

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